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EXISTENCE, UNIQUENESS AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO THE EQUATIONS OF VISCOELASTICITY WITH VOIDS*

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Abstract—This paper is concerned with the linear theroy of viscoelastic materials with voids. We study uniqueness, existence and asymptotic behaviour for the solutions of the dynamical problem. The uniqueness theorem is obtained by means of the power type function method. We use the semigroup theory of linear operators to obtain existence and continuous dependence of solutions. In the last section, we study the asymptotic behaviour of solutions. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Goodman and Cowin (1972) have introduced the concept of a distributed body as a continuum model for granular and porous bodies. In this theory, the bulk density ρ, matrix density γ and matrix volume fraction v are related by $\rho = \gamma v$. Using this concept Nunziato and Cowin (1979) have presented a theory for the behaviour of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material.

Cowin and Nunziato (1983) have obtained some results in the linear theory. A recent survey of this theory can be found in the book by Ciarletta and Ieşan (1993).

In this paper we consider the linear theory of viscoelasticity with voids which was studied in Ciarletta (1989) and Ciarletta and Scalia (1991) and we obtain some results for the dynamical problem.

First, we obtain uniqueness of solutions. It is suitable to say that our results apply for a class of problems that do not agree with the one presented in Ciarletta and Scalia (1991). Our approach is developed on the basis of the power type function method and can be considered a natural extension of the results of Chirita and Rionero (1991).

Second, we obtain an existence theorem for the homogeneous Dirichlet boundary conditions. To this end we use the semigroup theory of linear operators (see e.g. Pazy, 1983).

Third, we obtain an asymptotic behaviour of the solutions of this problem when load terms are zero. Our approach is inspired in the results of Dafermos (1976).

In Section 2 we state the basic equations and other preliminaries. Section 3 is devoted to state the uniqueness theorem. The existence result is stated in Section 4 and the asymptotic behaviour of solutions is presented in Section 5.

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2. BASIC EQUATIONS

We consider a body that at time t_0 occupies a regular region Ω_0 of the three-dimensional Euclidean space. Let $\partial\Omega_0$ be the boundary of Ω_0. We refer the motion to the reference configuration Ω_0 and a fixed system of rectangular Cartesian axes Ox_k $(k = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions. We also use a superposed dot to denote partial differentiation with respect to the time. As usual, letters in boldface stand for tensors of order $p > 1$, and, if $\mathbf{v}$ has order p, we write $v_{i_1, i_2, \ldots, i_p}$ for its components in the Cartesian coordinate frame.

The basic equations for the linear theory of viscoelastic solids with voids are given by the evolutive equalities (see Cowin and Nunziato, 1983)

$$t_{ij,j} + f_i = \rho \ddot{u}_i,$$
$$h_{j,j} + g + L = \rho\chi\ddot{\phi}, \tag{1}$$

the constitutive equations (see Ciarletta and Scalia, 1991)

$$t_{ij} = \int_{-\infty}^{t} [G_{ijrs}(t-s)\dot{e}_{rs}(s) + B_{ij}(t-s)\dot{\phi}(s) + D_{ijr}(t-s)\dot{\phi}_{,r}(s)]\, \mathrm{d}s,$$

$$h_i = \int_{-\infty}^{t} [D_{rsi}(t-s)\dot{e}_{rs}(s) + D_i(t-s)\dot{\phi}(s) + A_{ij}(t-s)\dot{\phi}_{,j}(s)]\, \mathrm{d}s,$$

$$g = -\int_{-\infty}^{t} [B_{ij}(t-s)\dot{e}_{ij}(s) + b(t-s)\dot{\phi}(s) + D_i(t-s)\dot{\phi}_{,i}(s)]\, \mathrm{d}s, \tag{2}$$

and the geometrical equations

$$e_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}), \tag{3}$$

on Ω_0. Here t_{ji} is the stress tensor, f_i is the body force per unit volume, ρ is the density in the reference configuration, $\mathbf{u} = (u_i)$ is the displacement vector, h_i is the equilibrated stress vector, g is the intrinsic equilibrated body force, L is the extrinsic equilibrated body force, ϕ is the change in the volume fraction, χ is the equilibrated inertia. The constitutive relaxation functions satisfy the symmetry relations

$$G_{ijrs} = G_{rsij}, \quad D_{ijr} = D_{jir}, \quad A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}. \tag{4}$$

We must recall that G_{ijrs}, B_{ij}, D_{ijr}, D_i, A_{ij} and b are functions of v_0, the reference volume fraction, and its gradient $(v_0)_{,i}$.

If the material symmetry is of a type that possesses a centre of symmetry, then the tensors of odd order, D_{ijk} and D_i, are null. If, in addition, the material is isotropic then

$$G_{ijkl}(s) = \lambda(s)\delta_{ij}\delta_{kl} + \mu(s)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$
$$A_{ij}(s) = A(s)\delta_{ij}, \quad B_{ij}(s) = B(s)\delta_{ij},$$

λ, μ, A and B are functions of v_0. The assumption of isotropy implies the vanishing of the gradient of v_0. In this case, the constitutive eqns (2) become

$$t_{ij} = \int_{-\infty}^{t} [\lambda(t-s)\delta_{ij}\dot{e}_{rr}(s) + 2\mu(t-s)\dot{e}_{ij}(s) + B(t-s)\delta_{ij}\dot{\phi}(s)]\, \mathrm{d}s,$$

$$h_i = \int_{-\infty}^{t} A(t-s)\dot{\phi}_{,i}(s)\, \mathrm{d}s,$$

$$g = -\int_{-\infty}^{t} [B(t-s)\dot{e}_{rr}(s) + b(t-s)\dot{\phi}(s)]\,\mathrm{d}s. \tag{5}$$

We emphasize that the constitutive coefficients λ, μ, A, B and b in the isotropic situation still depend on v_0, the volume fraction in the reference configuration.

If we substitute the constitutive eqns (2) into the motion equations, we obtain the evolution equations for anisotropic materials:

$$\rho\ddot{u}_i = f_i + \left[\int_{-\infty}^{t} [G_{ijrs}(t-s)\dot{e}_{rs}(s) + B_{ij}(t-s)\dot{\phi}(s) + D_{ijr}(t-s)\dot{\phi}_{,r}(s)]\,\mathrm{d}s\right]_{,j},$$

$$\rho\chi\ddot{\phi} = L + \left[\int_{-\infty}^{t} [D_{rsi}(t-s)\dot{e}_{rs}(s) + D_i(t-s)\dot{\phi}(s) + A_{ij}(t-s)\dot{\phi}_{,j}(s)]\,\mathrm{d}s\right]_{,i},$$

$$-\int_{-\infty}^{t} [B_{ij}(t-s)\dot{e}_{ij}(s) + b(t-s)\dot{\phi}(s) + D_i(t-s)\dot{\phi}_{,i}(s)]\,\mathrm{d}s. \tag{6}$$

With the notations

$$t_{kl}n_k = t_l, \quad h_k n_k = h,$$

a set of mixed boundary conditions can be expressed in the form

$$u_i = \tilde{u}_i \quad \text{on}\ \partial\Omega_0^{\mathrm{u}}, \quad t_l = \tilde{t}_l \quad \text{on}\ \partial\Omega_0^{\mathrm{t}} = \partial\Omega_0 - \partial\Omega_0^{\mathrm{u}},$$

$$\phi = \tilde{\phi} \quad \text{on}\ \partial\Omega_0^{\phi}, \quad h = \tilde{h} \quad \text{on}\ \partial\Omega_0^{h} = \partial\Omega_0 - \partial\Omega_0^{\phi}. \tag{7}$$

We adjoin the initial conditions

$$\mathbf{u}(\mathbf{X}, -s) = \mathbf{z}^0(\mathbf{X}, s), \quad \dot{\mathbf{u}}(\mathbf{X}, 0) = \mathbf{v}^0(\mathbf{X}),$$

$$\phi(\mathbf{X}, -s) = \alpha^0(\mathbf{X}, s), \quad \dot{\phi}(\mathbf{X}, 0) = \psi^0(\mathbf{X}). \tag{8}$$

3. UNIQUENESS RESULTS

In Ciarletta and Scalia (1991) have presented a uniqueness theorem for the linear theory of viscoelastic solids with voids. Their results derive from the positivity of a certain quadratic form. In this section we obtain a uniqueness result without this assumption, but we impose the positivity of the dissipation. Thus, the results we present here can be applied to a different class of problems.

Let $\mathscr{U}$, $\mathscr{I}$, $\mathscr{D}$, $\mathscr{K}$ and $\mathscr{W}$ be the functions defined by

$$\begin{aligned}\mathscr{U}(t) = \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{\Omega_0} &(G_{ijrs}(2t-\tau-z)\dot{e}_{ij}(\tau)\dot{e}_{rs}(z)\\ &+ A_{ij}(2t-\tau-z)\dot{\phi}_{,i}(\tau)\dot{\phi}_{,j}(z)\\ &+ b(2t-\tau-z)\dot{\phi}(\tau)\dot{\phi}(z)\\ &+ D_{ijr}(2t-\tau-z)(\dot{e}_{ij}(\tau)\dot{\phi}_{,r}(z) + \dot{e}_{ij}(z)\dot{\phi}_{,r}(\tau))\\ &+ D_k(2t-\tau-z)(\dot{\phi}(\tau)\dot{\phi}_{,r}(z) + \dot{\phi}(z)\dot{\phi}_{,r}(\tau))\\ &+ B_{ij}(2t-\tau-z)(\dot{e}_{ij}(\tau)\dot{\phi}(z) + \dot{e}_{ij}(z)\dot{\phi}(\tau)))\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}z,\end{aligned}$$

$$\mathscr{I}(t) = \frac{1}{2}\int_{-\infty}^{2t}\int_{-\infty}^{0}\int_{\Omega_0} (G_{ijrs}(2t-\tau-z)\dot{e}_{ij}(\tau)\dot{e}_{rs}(z)$$

$$
\begin{aligned}
&+A_{ij}(2t-\tau-z)\dot{\phi}_{,i}(\tau)\dot{\phi}_{,j}(z)\\
&+b(2t-\tau-z)\dot{\phi}(\tau)\dot{\phi}(z)\\
&+D_{ijr}(2t-\tau-z)(\dot{e}_{ij}(\tau)\dot{\phi}_{,r}(z)+\dot{e}_{ij}(z)\dot{\phi}_{,r}(\tau))\\
&+D_k(2t-\tau-z)(\dot{\phi}(\tau)\dot{\phi}_{,r}(z)+\dot{\phi}(z)\dot{\phi}_{,r}(\tau))\\
&+B_{ij}(2t-\tau-z)(\dot{e}_{ij}(\tau)\dot{\phi}(z)+\dot{e}_{ij}(z)\dot{\phi}(\tau)))\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}z,
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{D}(t) = \int_0^t\int_{-\infty}^s\int_{-\infty}^s\int_{\Omega_0}&(\dot{G}_{ijrs}(2s-\tau-z)\dot{e}_{ij}(\tau)\dot{e}_{rs}(z)\\
&+\dot{A}_{ij}(2s-\tau-z)\dot{\phi}_{,i}(\tau)\dot{\phi}_{,j}(z)\\
&+\dot{b}(2s-\tau-z)\dot{\phi}(\tau)\dot{\phi}(z)\\
&+\dot{D}_{ijr}(2s-\tau-z)(\dot{e}_{ij}(\tau)\dot{\phi}_{,r}(z)+\dot{e}_{ij}(z)\dot{\phi}_{,r}(\tau))\\
&+\dot{D}_k(2s-\tau-z)(\dot{\phi}(\tau)\dot{\phi}_{,r}(z)+\dot{\phi}(z)\dot{\phi}_{,r}(\tau))\\
&+\dot{B}_{ij}(2s-\tau-z)(\dot{e}_{ij}(\tau)\dot{\phi}(z)+\dot{e}_{ij}(z)\dot{\phi}(\tau)))\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}z\,\mathrm{d}s,\\
\mathscr{K}(t) = \frac{1}{2}\int_{\Omega_0}&(\rho\dot{u}_i\dot{u}_i+\rho\chi\dot{\phi}\dot{\phi})\,\mathrm{d}v,\\
\mathscr{W}(t) = \frac{1}{2}\int_{\Omega_0}&(\rho\dot{u}_i(0)\dot{u}_i(2t)+\rho\chi\dot{\phi}(0)\dot{\phi}(2t))\,\mathrm{d}v.
\end{aligned}
\tag{9}
$$

Theorem 3.1. Assume that relations (4) are satisfied. Let

$$
\mathscr{L}(r,s) = \int_{\Omega_0}(\rho f_i(r)\dot{u}_i(s)+\rho\chi L(r)\dot{\phi}(s))\,\mathrm{d}v+\int_{\partial\Omega_0}(t_i(r)\dot{u}_i(s)+h(r)\dot{\phi}(s))\,\mathrm{d}s,
$$

for all r, $s\in[0,\infty)$. Then

$$
\mathscr{U}(t)-\mathscr{K}(t) = \frac{1}{2}\int_0^t[\mathscr{L}(t+s,t-s)-\mathscr{L}(t-s,t+s)]\,\mathrm{d}s+\mathscr{I}(t)-\mathscr{W}(t).
$$

Proof: First, we introduce the notation

$$
\mathscr{H}(r,s) = t_{ji}(r)\dot{e}_{ji}(s)+h_i(r)\dot{\phi}_i(s)-g(r)\dot{\phi}(s).
$$

After some calculations we conclude

$$
\begin{aligned}
\mathscr{H}(t-s,t+s)-\mathscr{H}(t+s,t-s) = \frac{\mathrm{d}}{\mathrm{d}s}\Bigg[\int_{-\infty}^{t-s}\int_{-\infty}^{t+s}&(G_{ijrs}(2t-\tau-z)\dot{e}_{ij}(\tau)\dot{e}_{rs}(z)\\
&+A_{ij}(2t-\tau-z)\dot{\phi}_{,i}(\tau)\dot{\phi}_{,j}(z)\\
&+b(2t-\tau-z)\dot{\phi}(\tau)\dot{\phi}(z)\\
&+D_{ijr}(2t-\tau-z)(\dot{e}_{ij}(\tau)\dot{\phi}_{,r}(z)+\dot{e}_{ij}(z)\dot{\phi}_{,r}(\tau))\\
&+D_k(2t-\tau-z)(\dot{\phi}(\tau)\dot{\phi}_{,r}(z)+\dot{\phi}(z)\dot{\phi}_{,r}(\tau))\\
&+B_{ij}(2t-\tau-z)(\dot{e}_{ij}(\tau)\dot{\phi}(z)+\dot{e}_{ij}(z)\dot{\phi}(\tau)))\,\mathrm{d}\tau\,\mathrm{d}z\Bigg].
\end{aligned}
\tag{10}
$$

From the evolution equations we obtain

$$\mathscr{H}(r,s) = [t_{ji}(r)\dot{u}_i(s) + h_j(r)\dot{\phi}(s)]_{,j} - [\rho\ddot{u}_i(r) - f_i(r)]\dot{u}_i(s) - [\rho\chi\ddot{\phi}(r) - L(r)]\dot{\phi}(s).$$

The previous equality leads to the relation

$$\int_{\Omega_0} \mathscr{H}(t-s,t+s)\,\mathrm{d}v = \mathscr{L}(t-s,t+s) + \frac{\mathrm{d}}{\mathrm{d}s}\left[\int_{\Omega_0} (\rho\dot{u}_i(t-s)\dot{u}_i(t+s) + \rho\chi\dot{\phi}(t-s)\dot{\phi}(t+s))\,\mathrm{d}v\right] - \int_{\Omega_0} (\rho\dot{u}_i(t-s)\ddot{u}_i(t+s) + \rho\chi\dot{\phi}(t-s)\ddot{\phi}(t+s))\,\mathrm{d}v.$$

After integration from 0 to t, we get

$$\int_0^t\int_{\Omega_0} [\mathscr{H}(t-s,t+s) - \mathscr{H}(t+s,t-s)]\,\mathrm{d}v\,\mathrm{d}s = -2\mathscr{K}(t) + \int_0^t [\mathscr{L}(t-s,t+s) - \mathscr{L}(t+s,t-s)]\,\mathrm{d}s + \int_{\Omega_0} (\rho\dot{u}_i(0)\dot{u}_i(2t) + \rho\chi\dot{\phi}(0)\dot{\phi}(2t))\,\mathrm{d}v. \quad (11)$$

Now, from (10) we deduce

$$\int_0^t\int_{\Omega_0} [\mathscr{H}(t-s,t+s) - \mathscr{H}(t+s,t-s)]\,\mathrm{d}v\,\mathrm{d}s = 2\mathscr{I}(t) - 2\mathscr{U}(t). \quad (12)$$

Relations (11) and (12) imply the desired result.

Theorem 3.2. Assume that relations (4) are satisfied. Let $\mathscr{P}(t) = \mathscr{L}(t,t)$, then

$$2\mathscr{U}(t) = \mathscr{U}(0) + \mathscr{K}(0) + \mathscr{D}(t) + \mathscr{I}(t) - \mathscr{W}(t) + \int_0^t \mathscr{P}(s)\,\mathrm{d}s - \frac{1}{2}\int_0^t [\mathscr{L}(t-s,t+s) - \mathscr{L}(t+s,t-s)]\,\mathrm{d}s, \quad (13)$$

and

$$2\mathscr{K}(t) = \mathscr{U}(0) + \mathscr{K}(0) + \mathscr{D}(t) - \mathscr{I}(t) + \mathscr{W}(t) + \int_0^t \mathscr{P}(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t [\mathscr{L}(t-s,t+s) - \mathscr{L}(t+s,t-s)]\,\mathrm{d}s, \quad (14)$$

for all $t \in [0,\infty)$.

Proof: Direct calculation shows that

$$\int_{\Omega_0} \mathscr{H}(t,t)\,\mathrm{d}v = \dot{\mathscr{U}}(t) - \dot{\mathscr{D}}(t).$$

We also have

$$\int_{\Omega_0} \mathscr{H}(t,t)\,\mathrm{d}v = \mathscr{P}(t) - \dot{\mathscr{K}}(t).$$

From the two previous relations we deduce the equality

$$\mathscr{U}(t)+\mathscr{K}(t) = \mathscr{U}(0)+\mathscr{K}(0)+\int_0^t \mathscr{P}(s)\,\mathrm{d}s+\mathscr{D}(t).$$

Theorem 3.1 and the former relation imply the desired result.

Now, we are ready to obtain the uniqueness result which is the object of this section.

Theorem 3.3. Assume that:

(a) The relations (4) are satisfied
(b) $\rho > 0, \chi > 0$
(c) The following inequality

$$\int_{-\infty}^{s}\int_{-\infty}^{s} (\dot{G}_{ijrs}(2s-\tau-z)e_{ij}(\tau)e_{rs}(z)+\dot{A}_{ij}(2s-\tau-z)\psi_i(\tau)\psi_j(z)+\dot{b}(2s-\tau-z)\phi(\tau)\phi(z)$$
$$+\dot{D}_{ijr}(2s-\tau-z)(e_{ij}(\tau)\psi_r(z)+e_{ij}(z)\psi_r(\tau))+\dot{D}_k(2s-\tau-z)(\phi(\tau)\psi_k(z)+\phi(z)\psi_k(\tau))$$
$$+\dot{B}_{ij}(2s-\tau-z)(e_{ij}(\tau)\phi(z)+e_{ij}(z)\phi(\tau)))\,\mathrm{d}\tau\,\mathrm{d}z \leqslant 0,$$

holds for every symmetric tensor e_{ij}, vector field ψ_i and scalar field ϕ in C^0 on $(-\infty, \infty)$.

Then, the problem determined by the eqns (1) and (2), the boundary conditions (7) and the initial conditions (8) has at most one solution.

Proof: Suppose that there are two solutions. We designate $(\bar{u}_i, \bar{\phi})$ the difference corresponding to the null data. From relation (14) we have

$$\int_{\Omega_0} (\rho\dot{\bar{u}}_i\dot{\bar{u}}_i+\rho\chi\dot{\bar{\phi}}^2)\,\mathrm{d}v-\bar{\mathscr{D}}(t) = 0.$$

Thus, we conclude that $\bar{u}_i = 0$, $\bar{\phi} = 0$.

Remark 3.1. This uniqueness result does not assume any condition on the positivity of the energy.

4. THE EXISTENCE THEOREM

In this section we use some results of the semigroup theory of linear operators (see e.g. Pazy, 1983) to obtain an existence theorem for the equations of the linear theory of viscoelastic solids with voids. We restrict our attention to the homogeneous boundary conditions

$$\mathbf{u} = 0, \quad \phi = 0, \quad \text{on } \partial\Omega_0 \times [0, \infty). \tag{15}$$

Remark 4.1. Our boundary conditions are usual in several kinds of studies on elastic materials with voids (Rusu, 1987; Martínez and Quintanilla, 1995), since ϕ is the change of the volume fraction in respect of a certain reference volume fraction v_0.

In the remainder of the paper we assume:

(i) The relations (4) are satisfied.
(ii) The mass density ρ and the equilibrated inertia χ are strictly positive.
(iii) There is a positive constant c_0 such that

$$\int_{\Omega_0} (G_{ijrs}(\infty)u_{i,j}u_{r,s} + A_{ij}(\infty)\phi_{,j}\phi_{,i} + b(\infty)\phi^2 + 2D_{ijr}(\infty)u_{i,j}\phi_{,r}$$

$$+2D_r(\infty)\phi\phi_{,r} + 2B_{ij}(\infty)u_{i,j}\phi)\,\mathrm{d}v \geqslant c_0 \int_{\Omega_0} (u_{i,j}u_{i,j} + \phi_{,i}\phi_{,i} + \phi^2)\,\mathrm{d}v, \quad (16)$$

for all $\mathbf{u} \in [C_0^\infty(\Omega_0)]^3$ and $\phi \in C_0^\infty(\Omega_0)$.

(iv) There is a positive function $\delta(s)$ such that

$$\int_{\Omega_0} (\ddot{G}_{ijrs}(s)u_{i,j}u_{r,s} + \ddot{A}_{ij}(s)\phi_{,i}\phi_{,j} + \ddot{b}(s)\phi^2 + 2\ddot{D}_{ijr}(s)u_{i,j}\phi_{,r} + 2\ddot{D}_k(s)\phi\phi_{,r} + 2\ddot{B}_{ij}(s)u_{i,j}\phi)\,\mathrm{d}v$$

$$\geqslant \delta(s) \int_{\Omega_0} (u_{i,j}u_{i,j} + \phi_{,r}\phi_{,r} + \phi^2)\,\mathrm{d}v, \quad (17)$$

for all $\mathbf{u} \in [C_0^\infty(\Omega_0)]^3$ and $\phi \in C_0^\infty(\Omega_0)$.

(v)

$$\dot{G}_{ijrs}(\infty) = \dot{A}_{ij}(\infty) = \dot{b}(\infty) = \dot{D}_{ijr}(\infty) = \dot{D}_k(\infty) = \dot{B}_{ij}(\infty) = 0. \quad (18)$$

Remark 4.2. The mechanical interpretation of the conditions on ρ and χ is obvious. Assumption (16) is natural for the elasticity with voids (see Martínez and Quintanilla, 1995). Assumption (17) is the extension to the viscoelasticity with voids of the usual convexity conditions for the viscoelastic theories (see Navarro, 1978a, b).

Remark 4.3. Assumption (16) and (17) are usually imposed on the symmetric part of the gradient of the deformation (3). Because of the boundary conditions and the Korn's first inequality our conditions are also satisfied.

Remark 4.4. Assumptions (iv) and (v) imply that there exists a positive function $\delta_1(s)$ such that

$$-\int_{\Omega_0} (\dot{G}_{ijrs}(s)u_{i,j}u_{r,s} + \dot{A}_{ij}(s)\phi_{,i}\phi_{,j} + 2\dot{D}_{ijr}(s)u_{i,j}\phi_{,r} + 2\dot{D}_k(s)\phi\phi_{,r}$$

$$+2\dot{B}_{ij}(s)u_{i,j}\phi + \dot{b}(s)\phi^2)\,\mathrm{d}v \geqslant \delta_1(s) \int_{\Omega_0} (u_{i,j}u_{i,j} + \phi_{,r}\phi_{,r} + \phi^2)\,\mathrm{d}v, \quad (19)$$

for all $\mathbf{u} \in [C_0^\infty(\Omega_0)]^3$ and $\phi \in C_0^\infty(\Omega_0)$. This inequality will be used later to deduce a positive energy.

An alternative form for the constitutive eqns (2) allows us to express the system (6) in the following form

$$\rho\ddot{u}_i = f_i + \left[G_{ijrs}(0)e_{rs} + B_{ij}(0)\phi + D_{ijr}(0)\phi_{,r} \right.$$

$$\left. + \int_0^\infty [\dot{G}_{ijrs}(s)e_{rs}(t-s) + \dot{B}_{ij}(s)\phi(t-s) + \dot{D}_{ijr}(s)\phi_{,r}(t-s)]\,\mathrm{d}s \right]_{,j},$$

$$\rho\chi\ddot{\phi} = L + \left[D_{rsi}(0)e_{rs} + D_i(0)\phi + A_{ij}(0)\phi_{,j} + \int_0^\infty [\dot{D}_{rsi}(s)e_{rs}(t-s) \right.$$

$$\left. + \dot{D}_i(s)\phi(t-s) + \dot{A}_{ij}(s)\phi_{,j}(t-s)]\,\mathrm{d}s \right]_{,i} - B_{ij}(0)e_{ij} - b(0)\phi - D_i(0)\phi_{,i}$$

$$-\int_0^\infty [\dot{B}_{ij}(s)e_{ij}(t-s) + \dot{b}(s)\phi(t-s) + \dot{D}_i(s)\phi_{,i}(t-s)]\,\mathrm{d}s. \quad (20)$$

We now transform the boundary-initial-value problem defined by eqns (20), initial conditions (8) and boundary conditions (15) into an abstract problem on a suitable Hilbert space.

Let ω be

$$\omega=(\mathbf{u},\mathbf{v},\phi,\psi,\mathbf{z},\alpha),$$

where $\mathbf{v}=\dot{\mathbf{u}}$, $\psi=\dot{\phi}$, $\mathbf{z}(s)=\mathbf{u}(t-s)$ and $\alpha(s)=\phi(t-s)$.

We denote

$$\mathscr{X}_0=\{(\mathbf{u},\mathbf{v},\phi,\psi,\mathbf{z},\alpha)\,;\mathbf{u}\in[C_0^\infty(\Omega_0)]^3,\mathbf{v}\in[C_0^\infty(\Omega_0)]^3,$$
$$\phi\in C_0^\infty(\Omega_0),\psi\in[C_0^\infty(\Omega_0)]^3,\mathbf{z}\in\mathscr{H}_1,\alpha\in\mathscr{H}_2\},$$

where

$$\mathscr{H}_1=[C_0^\infty([0,\infty),\mathbf{W}_0^{1,2}(\Omega_0)\cap\mathbf{W}^{2,2}(\Omega_0))]^3,$$
$$\mathscr{H}_2=C_0^\infty([0,\infty),\mathbf{W}_0^{1,2}(\Omega_0)\cap\mathbf{W}^{2,2}(\Omega_0)),$$

here $\mathbf{W}_0^{1,2}(\Omega_0)=[W_0^{1,2}(\Omega_0)]^3$, and $\mathbf{W}^{2,2}(\Omega_0)=[W_0^{2,2}(\Omega_0)]^3$ being $W_0^{1,2}(\Omega_0)$, and $W_0^{2,2}(\Omega_0)$ the usual Sobolev spaces (see e.g. Adams, 1975).

Now, we consider $\mathscr{X}$ the completion of $\mathscr{X}_0$ with respect to the norm induced by the inner product

$$\begin{aligned}\langle(\mathbf{u},\mathbf{v},\phi,\psi,\mathbf{z},\alpha),(\mathbf{u}^*,\mathbf{v}^*,\phi^*,\psi^*,\mathbf{z}^*,\alpha^*)\rangle &= \int_{\Omega_0}(G_{ijrs}(\infty)u_{i,j}u^*_{r,s}+A_{ij}(\infty)\phi^*_{,j}\phi_{,i}+b(\infty)\phi\phi^*\\
&+D_{ijr}(\infty)(u_{i,j}\phi^*_{,r}+u^*_{i,j}\phi_{,r})+D_r(\infty)(\phi\phi^*_{,r}+\phi^*\phi_{,r})\\
&+B_{ij}(\infty)(u_{i,j}\phi^*+u^*_{i,j}\phi)+\rho v_iv_i^*+\rho\chi\psi\psi^*)\,\mathrm{d}v\\
&-\int_{\Omega_0}\int_0^\infty(\dot{G}_{ijrs}(s)(u_{i,j}-z_{i,j})(u^*_{r,s}-z^*_{r,s})+\dot{A}_{ij}(s)(\phi^*_{,j}-\alpha^*_{,j})(\phi_{,i}-\alpha_{,i})+\dot{b}(s)(\phi-\alpha)(\phi^*-\alpha^*)\\
&+\dot{D}_r(s)((\phi-\alpha)(\phi^*_{,r}-\alpha^*_{,r})+(\phi^*-\alpha^*)(\phi_{,r}-\alpha_{,r}))\\
&+\dot{D}_{ijr}(s)((u_{i,j}-z_{i,j})(\phi^*_{,r}-\alpha^*_{,r})+(u^*_{i}-z^*_{i})(\phi_{,r}-\alpha_{,r}))\\
&+\dot{B}_{ij}(s)((u_{i,j}-z_{i,j})(\phi^*-\alpha^*)+(u^*_{i,j}-z^*_{i,j})(\phi-\alpha))\,\mathrm{d}s\,\mathrm{d}v.\end{aligned}\tag{21}$$

This inner product is well defined, because of the assumptions stated previously.

Let us consider the operators

$$B_i\mathbf{u}=\frac{1}{\rho}[G_{jirs}(0)u_{s,r}]_{,j},$$

$$C_i\phi=\frac{1}{\rho}[B_{ji}(0)\phi+D_{ijp}(0)\phi_{,p}]_{,j},$$

$$P_i\mathbf{z}=\frac{1}{\rho}\left[\int_0^\infty\dot{G}_{jirs}(s)z_{s,r}(s)\,\mathrm{d}s\right]_{,j},$$

$$E_i\alpha=\frac{1}{\rho}\left[\int_0^\infty(\dot{B}_{ij}(s)\alpha(s)+\dot{D}_{ijp}(s)\alpha_{,p}(s))\,\mathrm{d}s\right]_{,j},$$

$$F\mathbf{u} = \frac{1}{\rho\chi}([D_{pqi}(0)u_{p,q}]_{,i} - B_{ij}(0)u_{i,j}),$$

$$G\phi = \frac{1}{\rho\chi}([D_i(0)\phi + A_{ij}(0)\phi_{,j}]_{,i} - b(0)\phi - D_k(0)\phi_{,k}),$$

$$H\mathbf{z} = \frac{1}{\rho\chi}\int_0^\infty ([\dot{D}_{pqi}(s)z_{p,q}(s)]_{,i} - \dot{B}_{pq}(s)z_{p,q}(s))\,\mathrm{d}s,$$

$$J\alpha = \frac{1}{\rho\chi}\int_0^\infty ([\dot{D}_i(s)\alpha(s) + \dot{A}_{ij}(s)\alpha_{,j}(s)]_{,i} - \dot{b}(s)\alpha(s) - \dot{D}_i(s)\alpha_{,i}(s))\,\mathrm{d}s,$$

$$R_i\mathbf{z} = -\frac{\partial}{\partial s}z_i(s), \quad W\alpha = -\frac{\partial}{\partial s}\alpha(s).$$

Let $\mathscr{A}$ be the matrix operator with domain

$$\mathscr{D}(\mathscr{A}) = \{(\mathbf{u},\mathbf{v},\phi,\psi,\mathbf{z},\alpha)\in\mathscr{X}\,;\mathscr{A}(\mathbf{u},\mathbf{v},\phi,\psi,\mathbf{z},\alpha)\in\mathscr{X}, \quad \mathbf{z}(0) = \mathbf{u}, \alpha(0) = \phi\},$$

defined by

$$\mathscr{A} = \begin{bmatrix} 0 & \mathbf{Id} & 0 & 0 & 0 & 0 \\ \mathbf{B} & 0 & \mathbf{C} & 0 & \mathbf{P} & \mathbf{E} \\ 0 & 0 & 0 & \mathbf{Id} & 0 & 0 \\ F & 0 & G & 0 & H & J \\ 0 & 0 & 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & 0 & 0 & W \end{bmatrix},$$

where $\mathbf{B} = (B_i)$, $\mathbf{P} = (P_i)$, $\mathbf{C} = (C_i)$, $\mathbf{E} = (E_i)$, $\mathbf{R} = (R_i)$ and $\mathbf{Id}$ is the identity operator. It is clear that $\mathscr{D}(\mathscr{A})$ is a subset which is dense in $\mathscr{X}$.

The boundary-initial-value problem (20), (8), (15) can be transformed into the following abstract equation in the Hilbert space $\mathscr{X}$,

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \mathscr{A}\omega(t) + \mathscr{F}(t), \quad \omega(0) = \omega_0, \tag{22}$$

where

$$\mathscr{F} = (0, \rho^{-1}\mathbf{f}, 0, (\rho\chi)^{-1}L, 0, 0),$$

$$\omega_0 = (\mathbf{u}^0, \mathbf{v}^0, \phi^0, \psi^0, \mathbf{z}^0, \alpha^0).$$

Lemma 4.1. The operator $\mathscr{A}$ satisfies the property

$$\langle\mathscr{A}\omega, \omega\rangle \leqslant 0,$$

for any $\omega\in\mathscr{D}(\mathscr{A})$.

Proof: Let $\omega = (\mathbf{u},\mathbf{v},\phi,\psi,\mathbf{z},\alpha)\in\mathscr{D}(\mathscr{A})$. Using the Divergence Theorem and the boundary conditions we find that

$$
\begin{aligned}
\langle \mathscr{A}\omega,\omega\rangle = & \int_{\Omega_0} (G_{ijrs}(\infty)v_{i,j}u_{r,s}+A_{ij}(\infty)\psi_{,i}\phi_{,j}+b(\infty)\psi\phi \\
& +D_{ijp}(\infty)(v_{i,j}\phi_{,p}+u_{i,j}\psi_{,p})+D_p(\infty)(\psi\phi_{,p}+\phi\psi_{,p}) \\
& +B_{ij}(\infty)(v_{i,j}\phi+u_{i,j}\psi)-G_{ijrs}(0)v_{i,j}u_{r,s}+A_{ij}(0)\psi_{,i}\phi_{,j}+b(0)\psi\phi \\
& +D_{ijp}(0)(v_{i,j}\phi_{,p}+u_{i,j}\psi_{,p})+D_p(0)(\psi\phi_{,p}+\phi\psi_{,p}) \\
& +B_{ij}(0)(v_{i,j}\phi+u_{i,j}\psi))\,\mathrm{d}v \\
& -\int_0^\infty (\dot{G}_{ijrs}(s)z_{r,s}(s)v_{i,j}+\dot{A}_{ij}(s)\alpha_{,j}(s)\psi_{,i}+\dot{b}(s)\alpha(s)\psi \\
& +\dot{D}_{ijp}(s)(v_{i,j}\alpha_{,p}(s)+z_{i,j}(s)\psi_{,p})+\dot{D}_p(s)(\psi\alpha_{,p}(s)+\alpha(s)\psi_{,p}) \\
& +\dot{B}_{ij}(s)(z_{i,j}(s)\phi+v_{i,j}\alpha(s)))\,\mathrm{d}s)\,\mathrm{d}v \\
& -\int_{\Omega_0}\int_0^\infty (\dot{G}_{ijrs}(s)(v_{i,j}+\dot{z}_{i,j}(s))(u_{r,s}-z_{r,s}(s)) \\
& +\dot{A}_{ij}(s)(\psi_{,i}+\dot{\alpha}_{,i}(s))(\phi_{,j}-\alpha_{,j}(s))+\dot{b}(s)(\psi+\dot{\alpha}(s))(\phi-\alpha(s)) \\
& +\dot{D}_{ijp}(s)[(v_{i,j}+\dot{z}_{i,j}(s))(\phi_{,p}-\alpha_{,p}(s))+(u_{i,j}-z_{i,j}(s))(\psi_{,p}+\dot{\alpha}_{,p}(s))] \\
& +\dot{D}_p(s)[(\psi+\dot{\alpha}(s))(\phi_{,p}-\alpha_{,p}(s))+(\psi_{,p}+\dot{\alpha}_{,p}(s))(\phi-\alpha(s))] \\
& +\dot{B}_{ij}(s)[(v_{i,j}+\dot{z}_{i,j}(s))(\phi-\alpha(s))+(u_{i,j}-z_{i,j}(s))(\psi+\dot{\alpha}(s))])\,\mathrm{d}s\,\mathrm{d}v \\
= & -\frac{1}{2}\int_{\Omega_0}\int_0^\infty (\ddot{G}_{ijrs}(s)(u_{i,j}-z_{i,j}(s))(u_{r,s}-z_{r,s}(s)) \\
& +\ddot{A}_{ij}(s)(\phi_{,j}-\alpha_{,j}(s))(\phi_{,i}-\alpha_{,i}(s)) \\
& +\ddot{b}(s)(\phi-\alpha(s))^2+2\ddot{D}_{ijp}(s)(u_{i,j}-z_{i,j}(s))(\phi_{,p}-\alpha_{,p}(s)) \\
& +2\ddot{D}_p(s)(\phi-\alpha(s))(\phi_{,p}-\alpha_{,p}(s)) \\
& +2\ddot{B}_{ij}(s)(u_{i,j}-z_{i,j}(s))(\phi-\alpha(s)))\,\mathrm{d}s\,\mathrm{d}v. \qquad (23)
\end{aligned}
$$

Lemma 4.1 follows from the convexity assumption (17).

Lemma 4.2. The operator $\mathscr{A}$ satisfies the range condition

$$\text{Range }(\mathbf{Id}-\mathscr{A}) = \mathscr{X}.$$

Proof: Let $\omega^* = (\mathbf{u}^*, \mathbf{v}^*, \phi^*, \psi^*, \mathbf{z}^*, \alpha^*) \in \mathscr{X}$. The range condition is satisfied whenever the system

$$
\begin{aligned}
& \mathbf{u}-\mathbf{v} = \mathbf{u}^*, \quad \phi-\psi = \phi^*, \quad \mathbf{v}-(\mathbf{Bu}+\mathbf{C}\phi+\mathbf{Pz}+\mathbf{E}\alpha) = \mathbf{v}^*, \\
& \psi-(F\mathbf{u}+G\phi+H\mathbf{z}+J\alpha) = \psi^*, \quad \mathbf{z}-\mathbf{Rz} = \mathbf{z}^*, \quad \alpha-W\alpha = \alpha^*, \qquad (24)
\end{aligned}
$$

has a solution $\omega = (\mathbf{u}, \mathbf{v}, \phi, \psi, \mathbf{z}, \alpha) \in \mathscr{D}(\mathscr{A})$. From the last two equations we have

$$\mathbf{z}(s) = e^{-s}\left(\mathbf{u}+\int_0^s e^\tau \mathbf{z}^*(\tau)\,\mathrm{d}\tau\right), \quad \alpha(s) = e^{-s}\left(\phi+\int_0^s e^\tau \alpha^*(\tau)\,\mathrm{d}\tau\right). \qquad (25)$$

If we substitute the two first equations for (24) and the eqns (25) into the third and fourth equations in (24), we obtain

$$\mathscr{A}'\begin{pmatrix}\mathbf{u}\\ \phi\end{pmatrix} = \begin{pmatrix}\mathbf{B}' & \mathbf{C}'\\ F' & G'\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \phi\end{pmatrix} = \begin{pmatrix}\mathbf{m}\\ \mathbf{n}\end{pmatrix}, \tag{26}$$

where

$$B_i'\mathbf{u} = u_i - \frac{1}{\rho}\left[\left(G_{ijrs}(0) + \int_0^\infty \dot{G}_{ijrs}(s)e^{-s}\,\mathrm{d}s\right)u_{r,s}\right]_{,j},$$

$$C_i'\phi = -\frac{1}{\rho}\left[\left(B_{ij}(0) + \int_0^\infty \dot{B}_{ij}(s)e^{-s}\,\mathrm{d}s\right)\phi + \left(D_{ijp}(0) + \int_0^\infty \dot{D}_{ijp}(s)e^{-s}\,\mathrm{d}s\right)\phi_{,p}\right]_{,j},$$

$$F'\mathbf{u} = -\frac{1}{\rho\chi}\left\{\left[\left(D_{pqi}(0) + \int_0^\infty \dot{D}_{pqi}(s)e^{-s}\,\mathrm{d}s\right)u_{p,q}\right]_{,i} - \left(B_{ij}(0) + \int_0^\infty \dot{B}_{ij}(s)e^{-s}\,\mathrm{d}s\right)u_{i,j}\right\},$$

$$G'\phi = \phi - \frac{1}{\rho\chi}\left\{\left[\left(D_i(0) + \int_0^\infty \dot{D}_i(s)e^{-s}\,\mathrm{d}s\right)\phi + \left(A_{ij}(0) + \int_0^\infty \dot{A}_{ij}(s)e^{-s}\,\mathrm{d}s\right)\phi_{,j}\right]_{,i} - \left(b(0) + \int_0^\infty \dot{b}(s)e^{-s}\,\mathrm{d}s\right)\phi - \left(D_j(0) + \int_0^\infty \dot{D}_j(s)e^{-s}\,\mathrm{d}s\right)\phi_{,j}\right\},$$

$$m_i = u_i^* + v_i^* + \frac{1}{\rho}\int_0^\infty\int_0^s e^{\tau-s}[\dot{G}_{ijrl}(s)z^*_{z,l}(\tau) + \dot{B}_{ij}(s)\alpha^*(\tau) + \dot{D}_{ijl}(s)\alpha^*_{,l}(\tau)]_{,j}\,\mathrm{d}\tau\,\mathrm{d}s,$$

$$n = \phi^* + \psi^* + \frac{1}{\rho\chi}\int_0^\infty\int_0^s e^{\tau-s}([\dot{D}_{pqi}(s)z^*_{p,q}(\tau) + \dot{D}_i(s)\alpha^*(\tau) + \dot{A}_{ij}(s)\alpha^*_{,j}(\tau)]_{,i}$$
$$- \dot{B}_{ij}(s)z^*_{i,j}(\tau) - \dot{D}_i(s)\alpha^*_{,i}(\tau)\dot{b}(s)\alpha^*(\tau))\,\mathrm{d}\tau\,\mathrm{d}s, \quad \mathbf{B}' = (B_i'), \quad \mathbf{C}' = (C_i'), \quad \mathbf{m} = (m_i). \tag{27}$$

To study this system for the unknowns $\mathbf{u}$ and ϕ, we introduce the following bilinear form on $\mathbf{W}_0^{1,2}(\Omega_0) \times W_0^{1,2}(\Omega_0)$

$$\mathscr{R}[(\mathbf{u},\phi),(\hat{\mathbf{u}},\hat{\phi})] = \langle(\mathbf{B}'\mathbf{u} + \mathbf{C}'\phi, F'\mathbf{u} + G'\phi),(\rho\hat{\mathbf{u}},\rho\chi\hat{\phi})\rangle_{\mathbf{L}^2\times\mathbf{L}^2}.$$

It is easy to see that $\mathscr{R}$ is bounded. We note that

$$\mathscr{R}[(\mathbf{u},\phi),(\mathbf{u},\phi)] = \int_{\Omega_0}(\rho u_i u_i + \rho\chi\phi^2)\,\mathrm{d}v + \int_{\Omega_0}\int_0^\infty (e^{-s}[G_{ijrs}(s)u_{i,j}u_{r,s} + A_{ij}(s)\phi_{,i}\phi_{,j} + b(s)\phi^2$$
$$+ 2D_{ijr}(s)u_{i,j}\phi_{,r} + 2D_r(s)\phi\phi_{,r} + 2B_{ij}(s)u_{i,j}\phi]\,\mathrm{d}s)\,\mathrm{d}v,$$

so that $\mathscr{R}$ is coercive on $\mathbf{W}_0^{1,2}(\Omega_0) \times W_0^{1,2}(\Omega_0)$. An easy calculation shows that $(\mathbf{m}, n)$ lies in $\mathbf{W}^{-1,2}(\Omega_0) \times W^{-1,2}(\Omega_0)$. Hence the Lax–Milgram theorem implies the existence of a solution

$(\mathbf{u}, \phi)$ to the eqn (26). Now, we may also conclude the existence of $\mathbf{v} \in \mathbf{W}_0^{1,2}(\Omega_0)$, $\psi \in W_0^{1,2}(\Omega_0)$, $\mathbf{z}$ and α that solve the system (24).

The previous lemmas lead to the next theorem.

Theorem 4.1. The operator $\mathscr{A}$ generates a contractive semigroup in $\mathscr{X}$.

Proof: The proof follows from the Lumer–Phillips corollary to the Hille–Yosida theorem.

Remark 4.5. The assumption (17) implies that the semigroup is contractive. It is possible to prove that $\mathscr{A}$ generates a semigroup if we assume a relaxed condition. Let us suppose that there exists a positive constant λ such that

$$\int_{\Omega_0} \big(\tfrac{1}{2}(\ddot{G}_{ijrs}(s) - \lambda \dot{G}_{ijrs}(s))u_{i,j}u_{r,s} + \tfrac{1}{2}(\ddot{A}_{ij}(s) - \lambda \dot{A}_{ij}(s))\phi_{,i}\phi_{,j}$$
$$+ \tfrac{1}{2}(\ddot{b}(s) - \lambda \dot{b}(s))\phi^2 + (\ddot{D}_{ijr}(s) - \lambda \dot{D}_{ijr}(s))u_{i,j}\phi_{,r}$$
$$+ (\ddot{D}_k(s) - \lambda \dot{D}_k(s))\phi\phi_{,k} + (\ddot{B}_{ij}(s) - \lambda \dot{B}_{ij}(s))u_{i,j}\phi\big)\, dv \geqslant 0. \tag{28}$$

If we postulate that the previous inequality and the inequality (19) are satisfied we may conclude that $\mathscr{A}$ generates a quasi-contractive semigroup.

Theorem 4.2. Assume that

$$f_i, L \in C^1([0,\infty), L^2) \cap C^0([0,\infty), W_0^{1,2}),$$

and

$$\omega_0 \in \mathscr{D}(\mathscr{A}).$$

Then, there exists a unique solution $\omega(t) \in C^1([0,\infty), \mathscr{X})$, with values in $\mathscr{D}(\mathscr{A})$, to the boundary-initial-value problem (22).

Remark 4.6. Since the semigroup defined by the operator $\mathscr{A}$ is contractive, we have the estimate

$$\|\omega(t)\| \leqslant \|\omega_0\|_{\mathscr{X}} + \int_0^t (\|\mathbf{f}(s)\|_{\mathbf{L}^2} + \|L(s)\|_{L^2})\, ds,$$

which proves the continuous dependence of the solutions upon initial data and body forces.

This result has been obtained in Ciarletta and Scalia (1991).

Remark 4.7. Under the assumptions (i)–(v) the problem of the linear theory of viscoelastic materials with voids is well posed.

5. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS

In this section we study the asymptotic behaviour of solutions in the case that the external sources are zero.

A sufficient condition to prove asymptotic stability for semigroups of contractions (see Dafermos, 1976) is the precompactness of the orbits jointly with the fact that there are not any eigenvalue $i\lambda$, $\lambda \in \mathbb{R}$ in the closed subspace $\mathscr{E}$ generated by the elements $\omega \in \mathscr{D}(\mathscr{A})$ such that $\langle \mathscr{A}\omega, \omega \rangle = 0$.

We need the following result:

Lemma 5.1.

$$\mathscr{A}^{-1}0 = \{0\}.$$

Proof: Let $\omega = (\mathbf{u}, \mathbf{v}, \phi, \psi, \mathbf{z}, \alpha)$ be a solution of the system:

$$v_i = 0, \tag{29}$$

$$\left[\int_0^\infty [\dot{G}_{ijrs}(s)u_{r,s}(t-s)+\dot{B}_{ij}(s)\phi(t-s)+\dot{D}_{ijr}(s)\phi_{,r}(t-s)]\,\mathrm{d}s + G_{ijrs}(0)u_{r,s}+B_{ij}(0)\phi+D_{ijr}(0)\phi_{,r}\right]_{,j} = 0, \tag{30}$$

$$\psi = 0, \tag{31}$$

$$\left[\int_0^\infty [\dot{D}_{rsi}(s)u_{r,s}(t-s)+\dot{D}_i(s)\phi(t-s)+\dot{A}_{ij}(s)\phi_{,j}(t-s)]\,\mathrm{d}s+D_{rsi}(0)u_{r,s}+D_i(0)\phi+A_{ij}(0)\phi_{,j}\right]_{,i}$$
$$-\int_0^\infty [\dot{B}_{ij}(s)u_{i,j}(t-s)+\dot{b}(s)\phi(t-s)+\dot{D}_i(s)\phi_{,i}(t-s)]\,\mathrm{d}s-B_{ij}(0)u_{i,j}-b(0)\phi-D_i(0)\phi_{,i} = 0, \tag{32}$$

$$-\frac{\partial}{\partial s}z_i(s) = 0, \tag{33}$$

$$-\frac{\partial}{\partial s}\alpha(s) = 0. \tag{34}$$

Integrating the last two equations we obtain:

$$\mathbf{z}(s) = \mathbf{u}, \quad \alpha(s) = \phi.$$

If we substitute then in (30) and (32), after a quadrature, we can write:

$$[G_{ijrs}(\infty)u_{r,s}+B_{ij}(\infty)\phi+D_{ijr}(\infty)\phi_{,r}]_{,j} = 0, \tag{35}$$

$$[D_{rsi}(\infty)u_{r,s}+D_i(\infty)\phi+A_{ij}(\infty)\phi_{,j}]_{,i}-B_{ij}(\infty)u_{i,j}-b(\infty)\phi-D_i(\infty)\phi_{,i} = 0. \tag{36}$$

If we multiply equation (35) by u_i, equation (36) by ϕ, and integrate over Ω_0, we obtain:

$$\int_{\Omega_0} [G_{ijrs}(\infty)u_{r,s}u_{i,j}+2B_{ij}(\infty)u_{i,j}\phi + 2D_{ijr}(\infty)u_{i,j}\phi_{,r}+2D_i(\infty)\phi\phi_{,i}+A_{ij}(\infty)\phi_{,j}\phi_{,i}+b(\infty)\phi^2]\,\mathrm{d}v = 0.$$

By (16), we can conclude:

$$u_{i,j} = 0, \quad \phi = 0,$$

and therefore $u_i = 0$.

Lemma 5.2. The operator $\mathscr{A}$ has not any eigenvalue $i\lambda$, $\lambda \in \mathbb{R}$, in the closed subspace $\mathscr{E}$.

Proof: Let $\omega \in \mathscr{E}$, then $\langle \mathscr{A}\omega, \omega\rangle = 0$. As a consequence of (23) we have:

$$-\frac{1}{2}\int_{\Omega_0}\int_0^{\infty}(\ddot{G}_{ijrs}(s)(u_{i,j}-z_{i,j}(s))(u_{r,s}-z_{r,s}(s))+\ddot{A}_{ij}(s)(\phi_{,j}-\alpha_{,j}(s))(\phi_{,i}-\alpha_{,i}(s))$$
$$+\ddot{b}(s)(\phi-\alpha(s))^2+2\ddot{D}_{ijp}(s)(u_{i,j}-z_{i,j}(s))(\phi_{,p}-\alpha_{,p}(s))+2\ddot{D}_p(s)(\phi-\alpha(s))(\phi_{,p}-\alpha_{,p}(s))$$
$$+2\ddot{B}_{ij}(s)(u_{i,j}-z_{i,j}(s))(\phi-\alpha(s)))\,\mathrm{d}s\,\mathrm{d}v=0,$$

and, by (17), we deduce:

$$u_{i,j}-z_{i,j}(s)=0,\quad \phi-\alpha(s)=0. \tag{37}$$

Now, integrating the last two equations of the system $\mathscr{A}\omega=i\lambda\omega$, we can write:

$$z_k=e^{-i\lambda s}u_k,\quad \alpha=\mathrm{e}^{-i\lambda s}\phi,$$

if we substitute in (37) we obtain:

$$(1-\mathrm{e}^{-i\lambda s})u_{j,k}=0,\quad (1-\mathrm{e}^{-i\lambda s})\phi=0,$$

and, therefore, $u_i=0$ and $\phi=0$ if $\lambda\neq 0$. □

Remark 5.1. In the absence of viscosity effects, it is easy to show that the system $\mathscr{A}\omega=i\lambda\omega$ has non trivial solutions.

Lemma 5.3. The orbits are precompact.

Proof: Let us denote

$$\mathscr{M}=\{(\mathbf{u}^*,\mathbf{v}^*,\phi^*,\psi^*,\mathbf{z}^*,\alpha^*)\in\mathscr{D}(\mathscr{A});$$
$$\mathbf{z}^*\in\mathbf{L}^{\infty}(\mathbb{R}^+,\mathbf{W}^{2,2}(\Omega_0)),\alpha^*\in\mathbf{L}^{\infty}(\mathbb{R}^+,W^{2,2}(\Omega_0))\}$$

and let be $(\mathbf{u}(t),\mathbf{v}(t),\phi(t),\psi(t),\mathbf{z}(t,s),\alpha(t,s))$ a solution starting in the point $(\mathbf{u}^*,\mathbf{v}^*,\phi^*,\psi^*,\mathbf{z}^*,\alpha^*)\in\mathscr{M}$.

Then, for all $t\in\mathbb{R}^+$,

$$(\mathbf{u}(t),\mathbf{v}(t),\phi(t),\psi(t),\mathbf{z}(t,s),\alpha(t,s))\in\mathscr{D}(\mathscr{A}),$$
$$\|\mathscr{A}(\mathbf{u}(t),\mathbf{v}(t),\phi(t),\psi(t),\mathbf{z}(t,s),\alpha(t,s))\|\leqslant\|\mathscr{A}(\mathbf{u}^*,\mathbf{v}^*,\phi^*,\psi^*,\mathbf{z}^*,\alpha^*)\|,$$

because the semigroup is contractive. Thus,

$$\mathbf{v}\in\mathbf{C}(\mathbb{R}^+,\mathbf{W}_0^{1,2}(\Omega_0)),$$
$$\psi\in C(\mathbb{R}^+,\mathbf{W}_0^{1,2}(\Omega_0)),\quad \frac{\partial\mathbf{z}}{\partial s}\in\mathbf{C}(\mathbb{R}^+,\mathbf{W}_0^{1,2}(\Omega_0)\cap\mathbf{W}^{2,2}(\Omega_0)),$$
$$\frac{\partial\alpha}{\partial s}\in C(\mathbb{R}^+,W_0^{1,2}(\Omega_0)\cap W^{2,2}(\Omega_0)),$$

$$\left[\int_0^{\infty}[\dot{G}_{ijrs}(s)u_{r,s}(t-s)+\dot{B}_{ij}(s)\phi(t-s)+\dot{D}_{ijr}(s)\phi_{,r}(t-s)]\,\mathrm{d}s\right.$$
$$\left.+G_{ijrs}(0)u_{r,s}+B_{ij}(0)\phi+D_{ijr}(0)\phi_{,r}\right]_{,j}\in\mathbf{C}(\mathbb{R}^+,\mathbf{L}^2(\Omega_0)), \tag{38}$$

$$\left[\int_0^\infty [\dot{D}_{rsi}(s)u_{r,s}(t-s)+\dot{D}_i(s)\phi(t-s)+\dot{A}_{ij}(s)\phi_{,j}(t-s)]\,\mathrm{d}s+D_{rsi}(0)u_{r,s}+D_i(0)\phi+A_{ij}(0)\phi_{,j}\right]_{,i}$$

$$-\int_0^\infty [\dot{B}_{ij}(s)u_{i,j}(t-s)+\dot{b}(s)\phi(t-s)+\dot{D}_i(s)\phi_{,i}(t-s)]\,\mathrm{d}s$$

$$-B_{ij}(0)u_{i,j}-b(0)\phi-D_i(0)\phi_{,i}\in C(\mathbb{R}^+,L^2(\Omega_0)), \quad (39)$$

Using the standard Picard iteration scheme (see Dafermos, 1970) we obtain:

$$\|(\mathbf{u},\phi)\|_{\mathbf{W}_0^{2,2}\times W_0^{2,2}(\Omega_0)} \leqslant k\|\mathbf{Bu}+\mathbf{C}\phi+\mathbf{Pz}+\mathbf{E}\alpha, F\mathbf{u}+G\phi+H\mathbf{z}+J\alpha\|_{\mathbf{L}^2(\Omega_0)\times L^2(\Omega_0)},$$

where k is a positive constant. Thereby, $\mathbf{u}\in\mathbf{C}(\mathbb{R}^+,\mathbf{W}_0^{2,2})$, and $\phi\in C(\mathbb{R}^+,W_0^{2,2}(\Omega_0))$.

The Rellich–Kondrachov theorem allows us to conclude that the orbits starting in $\mathcal{M}$ are precompact. But, $\mathcal{M}$ is dense in $\mathcal{X}$ and closed (see Dafermos, 1974); therefore, any orbit starting in $\mathcal{X}$ is precompact. □

As a consequence of the previous lemmas we conclude:

Theorem 5.1. Let $(\mathbf{u}(t),\mathbf{v}(t),\phi(t),\psi(t),\mathbf{z}(t,s),\alpha(t,s))$ be the solution of (20) with external sources $f_i = L = 0$, initial conditions (8) and boundary conditions (15); then

$$\lim_{t\to\infty}(\mathbf{u}(t),\mathbf{v}(t),\phi(t),\psi(t),\mathbf{z}(t,s),\alpha(t,s)) = 0$$

in the norm induced by the scalar product (21).

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